TBA equations for excited states in the sine–Gordon model

2004 J. Phys. A: Math. Gen. 37 1903

(http://iopscience.iop.org/0305-4470/37/5/028)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 1903–1925 PII: S0305-4470(04)68972-4

TBA equations for excited states in the sine–Gordon model

János Balog and Árpád Hegedűs

Research Institute for Particle and Nuclear Physics, Hungarian Academy of Sciences, H-1525 Budapest 114, POB 49, Hungary

Received 12 September 2003
Published 19 January 2004
Online at stacks.iop.org/JPhysA/37/1903 (DOI: 10.1088/0305-4470/37/5/028)

Abstract

We propose thermodynamic Bethe ansatz (TBA) integral equations for multi-particle soliton (fermion) states in the sine–Gordon (massive Thirring) model. This is based on T-system and Y-system equations, which follow from the Bethe ansatz solution in the light-cone lattice formulation of the model. Even and odd charge sectors are treated on an equal footing, corresponding to periodic and twisted boundary conditions, respectively. The analytic properties of the Y-system functions are conjectured on the basis of the large volume solution of the system, which we find explicitly. A simple relation between the TBA Y-functions and the counting function variable of the alternative non-linear integral equation (Destri–deVega equation) description of the model is given. At the special value $\beta^2 = 6\pi$ of the sine–Gordon coupling, exact expressions for energy and momentum eigenvalues of one-particle states are found.

PACS numbers: 05.70.Ce, 03.70.+k, 11.55.−m

1. Introduction

A better theoretical understanding of finite size (FS) effects is one of the most important problems in quantum field theory (QFT). The study of FS effects is a useful method of analysing the structure of QFT models and it is an indispensable tool in the numerical simulation of lattice field theories.

Lüscher [1] derived a general formula for the FS corrections to particle masses in the large volume limit. This formula, which is generally applicable for any QFT model in any dimension, expresses the FS mass corrections in terms of an integral containing the forward scattering amplitude analytically continued to unphysical (complex) energy. It is most useful in (1+1)-dimensional integrable models [2], where the scattering data are available explicitly.

The usefulness of the study of the mass gap in finite volume is demonstrated [3] by the introduction of the Lüscher–Weisz–Wolff running coupling that enables the interpolation between the large volume (non-perturbative) and the small volume (perturbative) regions in both two-dimensional sigma models and QCD.

An important tool in the study of two-dimensional integrable field theories is the thermodynamic Bethe ansatz (TBA). This thermodynamical method was initiated by Yang and Yang [4] and allows the calculation of the free energy of the particle system. The calculation was applied to the XXZ model by Takahashi and Suzuki [5] who derived the TBA integral equations for the free energy starting from the Bethe ansatz solution of the system and using the 'string hypothesis' describing the distribution of Bethe roots. The form of the resulting TBA equations strongly depends on whether the anisotropy parameter p_0 (see section 2) is an integer, rational or irrational number.

The TBA equations also determine FS effects in relativistic (Euclidean) invariant two-dimensional field theory models where the free energy is related to the ground state energy in finite volume by a modular transformation interchanging spatial extension and (inverse) temperature. Zamolodchikov [6] initiated the study of TBA equations for two-dimensional integrable models by pointing out that TBA equations can also be derived starting from the (dressed) Bethe ansatz equations formulated directly in terms of the (infinite volume) scattering phase shifts of the particles. In this approach the FS dependence of the ground state energy has been studied [7] in many integrable models, mainly those formulated as perturbations of minimal conformal models.

TBA methods have been developed also in lattice statistical physics [8]. Actually the basic functional relations, Y-system equations, TBA integral equations and also other techniques playing an important role in the TBA analysis of continuum models were originally introduced here. TBA equations describing the FS energy of excited states were proposed for some models. Also non-linear integral equations, similar to the Destri–deVega equations, appeared here first. An important model, the tricritical Ising model perturbed by its $\phi_{1,3}$ operator, has been studied in detail [9]. The TBA equations describing all excited states are worked out in this example, both for the massive and the massless perturbed models.

The TBA description of excited states is less systematic in continuum models. The excited state TBA systems first studied [10, 11] are not describing particle states, they correspond to ground states in charged sectors of the model. An interesting suggestion is to obtain excited state TBA systems by analytically continuing [11] those corresponding to the ground state energy. TBA equations for scattering states were suggested for perturbed field theory models by the analytic continuation method [12]. Excited state TBA equations were also suggested for scattering multi-particle states for the sine–Gordon (SG) model at its $N = 2$ supersymmetric point [13].

The BLZ programme [14] has been initiated to derive functional relations, Y-systems and TBA integral equations, both for the ground state and for excited states, directly in the continuum. The construction is complete for the conformal field theory limit of the models only although some examples in massive perturbed models are also worked out. The original construction is based on the R-matrix corresponding to the quantum deformed loop algebra $U_q\left(A_1^{(1)}\right)$ and can be applied to minimal models perturbed by their $\phi_{1,3}$ field but it also works in the $U_q\left(A_2^{(2)}\right)$ case [15], which is relevant to the $\phi_{1,2}$ and $\phi_{2,1}$ perturbations. In this family, the TBA equations for the first two excited states in the non-unitary perturbed $\mathcal{M}_{3,5}$ model are worked out explicitly.

An alternative to the TBA equations is provided by the Destri–deVega (DdV) non-linear integral equations. They were originally suggested [16] for the ground state of the SG or the closely related massive Thirring (MT) model, but can be systematically generalized to also describe excited states; the (common) even charge sector of the SG/MT model [17], the odd charge sector in both SG and MT [18] and even the states of perturbed minimal models, which can be represented as restrictions of the SG model [19]. The advantage of this non-linear integral equation approach is that when available (SG/MT model and restrictions), it gives a systematic description of all excited states. The method has been generalized to higher rank imaginary coupling affine Toda models [20].

Meanwhile an alternative derivation of the TBA equations, which is independent of the validity of the string hypothesis, has been found in the lattice approach for the generalized Hubbard model [21] and also for the XXZ model [22]. Both models are solvable on the lattice by the Bethe ansatz method and the new derivation of the TBA equations proceeds via the introduction of T-system and Y-system functions satisfying functional relations which can be transformed into TBA-type integral equations. Instead of the string hypothesis used in the original derivation [5], here the basic assumption is about the analytic structure of the Y-system functions (distribution of their zeros) and the validity of these assumptions can be convincingly demonstrated numerically, at least for small lattices, by starting from the numerical solution of the Bethe ansatz equations. The Takahashi–Suzuki results for the ground state energy are reproduced and new TBA equations for the excited states are found in this way. Actually for the excited state problem the TBA equations are supplemented by quantization conditions restricting the allowed values of particle momenta, which is natural for particles confined in a box.

In this paper we borrow the ideas of the lattice approach and apply them to the continuum QFT case. Following the steps of the lattice construction, we systematically build the T-system and Y-system functions for all multi-soliton (multi-fermion) states of the SG (MT) model. We find the appropriate continuum TBA equations and quantization conditions and a link between the TBA and the DdV equations.

Our starting point is the light-cone lattice regularization [23] of the SG/MT model and the Bethe ansatz solution, which we briefly recall in section 2. We use twisted boundary conditions on the lattice and show that by changing the value of the parameter characterizing the boundary conditions we can describe both the even and the odd sectors of the model on an equal footing. (Previously only the even sector was treated in the light-cone lattice formulation and the properties of the states belonging to the odd sector were conjectured by postulating the corresponding DdV equation in the continuum limit [18].) In section 3 we introduce the T-system and Y-system functions and establish the link between the TBA Y-system and the 'counting function' of the DdV approach. This link is explicitly given by (18). In section 4 simple properties of the counting function are recalled. In section 5 we write down the light-cone lattice TBA equations together with the quantization conditions. In section 6 we take the (finite volume) continuum limit of the equations. Multi-particle energy and momentum expressions are given in section 7. In section 8 we discuss a special case ($p_0 = 4$) where we can find exact expressions for the one-particle energies and momenta. In section 9 we discuss the DdV equation and its analytic continuation to the entire complex rapidity plane. In section 10, using the link between the DdV equation and the TBA Y-system, we find the explicit solution of the infinite volume limit of the TBA problem.

To transform the Y-system equations into TBA integral equations (and quantization conditions) we need to know the analytic properties of the Y-system functions, in particular the distribution of their zeros. From the explicit solution found in section 10 we know this distribution in the infinite volume limit. Our main assumption in this paper is that the qualitative properties of this distribution remain the same for finite volume. Using this conjecture we can write down the complete set of equations determining multi-particle momenta and energies in the TBA approach. The ground state (no particles), one-particle and two-particle problems are discussed in detail in sections 11, 12 and 13, respectively. We have verified the validity of our main assumption by numerically comparing the results of the TBA approach with those obtained using the DdV equations. This is briefly described in section 14 and finally our conclusions are summarized in section 15. The technical details of the transformation of Y-system type functional relations into TBA type integral equations are discussed in the appendix.

For simplicity, in this paper we restrict attention to the repulsive case $p_0 > 2$ of the SG coupling and consider integer p_0 only. Moreover, we consider multi-soliton states only and no states containing both solitons and anti-solitons at the same time. We believe that appropriate TBA systems can also be found for more general couplings and more general states but the special cases we are considering in this paper are sufficient to present the main ideas and assumptions. Finally we note that although in the SG/MT case, the excited state TBA description is 'superfluous' since we already have the DdV equations to study FS physics, one can hope that the simple pattern of the excited state TBA systems we find here means that similar systems can also be found in other systems, where no DdV type alternative is available.

2. Light-cone approach to twisted SG model

Our starting point is the Bethe ansatz solution of the integrable lattice regularization of the sine–Gordon field theory [23]. Here we briefly summarize the results of this approach [16–18] to the sine–Gordon (and massive Thirring) model. The fields of the regularized theory are defined at sites ('events') of a light-cone lattice and the dynamics of the system is defined by translations in the left and right light-cone directions. These are given by transfer matrices of the six-vertex model with anisotropy γ and alternating inhomogeneities. This approach is particularly useful for calculating the finite size dependence of physical quantities. We take N points (N even) in the spatial direction and use twisted periodic boundary conditions. The lattice spacing is related to L, the (dimensionful) size of the system:

$$a = \frac{L}{N}. \tag{1}$$

The physical states of the system are characterized by the set of Bethe roots $\{w_j, j = 1, \ldots, m \leqslant \frac{N}{2}\}$, which satisfy the Bethe ansatz equations (BAE)

$$\frac{Q(w_j + 2\mathrm{i})}{Q(w_j - 2\mathrm{i})} = -\frac{T_0(w_j + \mathrm{i})}{T_0(w_j - \mathrm{i})} \qquad j = 1, \ldots, m. \tag{2}$$

Here

$$T_0(\xi) = \left[k\left(\xi - \frac{2\Theta}{\pi}\right)k\left(\xi + \frac{2\Theta}{\pi}\right)\right]^{N/2} \tag{3}$$

where Θ is the inhomogeneity parameter,

$$k(\xi) = \frac{\sinh\frac{\gamma}{2}\xi}{\sin\gamma} \tag{4}$$

and

$$Q(\xi) = \mathrm{e}^{\frac{\omega\xi}{2}}\prod_{j=1}^{m}\sinh\frac{\gamma}{2}(\xi - w_j) \tag{5}$$

where the parameter ω characterizes the twist of the boundary condition. We shall also use the parametrization

$$\gamma = \frac{\pi}{p_0} = \frac{\pi}{p+1}. \tag{6}$$

Finally the energy (E) and momentum (P) of the physical state can be obtained from the eigenvalues of the light-cone transfer matrices:

$$\mathrm{e}^{\mathrm{i}a(E\pm P)} = (-1)^m \,\mathrm{e}^{\pm 2\mathrm{i}\omega} \frac{Q\left[\pm\left(-\mathrm{i}+\frac{2\Theta}{\pi}\right)\right]}{Q\left[\pm\left(\mathrm{i}+\frac{2\Theta}{\pi}\right)\right]}. \tag{7}$$

Besides the usual procedure, taking the thermodynamic limit ($N \to \infty$) first, followed by the continuum limit ($a \to 0$) one can also study the continuum limit in finite volume by taking $N \to \infty$ and tuning the inhomogeneity parameter Θ simultaneously as

$$\Theta = \ln \frac{2}{\mathcal{M}a} = \ln \frac{2N}{l} \tag{8}$$

where the mass parameter $\mathcal{M}$ is the infinite volume physical mass of the sine–Gordon solitons and we have introduced $l = \mathcal{M}L$, the dimensionless size of the system.

3. T-system and Y-system

Besides $Q(\xi)$ and $T_0(\xi)$ it is useful to introduce [22] the (half-)infinite sequence of functions $T_s(\xi)$ for $s = -1, 0, 1, \ldots$. In the six-vertex model these are eigenvalues of the transfer matrices corresponding to higher spin representations (in the auxiliary space) and are obtained by the fusion procedure. Equivalently we can simply define them as $T_{-1}(\xi) = 0$ and

$$T_s(\xi) = Q(\xi + \mathrm{i} + \mathrm{i}s)Q(\xi - \mathrm{i} - \mathrm{i}s) \sum_{j=0}^{s} q[\xi + \mathrm{i}(2j - s)] \tag{9}$$

for $s = 0, 1, \ldots,$ where

$$q(\xi) = \frac{T_0(\xi)}{Q(\xi - \mathrm{i})Q(\xi + \mathrm{i})}. \tag{10}$$

It is obvious from the definition that the T_s are periodic functions,

$$T_s(\xi + 2\mathrm{i}p_0) = T_s(\xi). \tag{11}$$

It can also be verified that they satisfy the T-system equations [22]

$$T_s(\xi + \mathrm{i})T_s(\xi - \mathrm{i}) = T_{s+1}(\xi)T_{s-1}(\xi) + T_0(\xi + \mathrm{i} + \mathrm{i}s)T_0(\xi - \mathrm{i} - \mathrm{i}s) \qquad s = 0, 1, \ldots \tag{12}$$

which can then be used to recursively calculate them starting from $T_1(\xi)$.

The next step is to introduce the functions

$$Y_s(\xi) = \frac{T_{s+1}(\xi)T_{s-1}(\xi)}{T_0(\xi + \mathrm{i} + \mathrm{i}s)T_0(\xi - \mathrm{i} - \mathrm{i}s)} \tag{13}$$

or equivalently

$$1 + Y_s(\xi) = \frac{T_s(\xi + \mathrm{i})T_s(\xi - \mathrm{i})}{T_0(\xi + \mathrm{i} + \mathrm{i}s)T_0(\xi - \mathrm{i} - \mathrm{i}s)}. \tag{14}$$

As a consequence of the T-system equations (12) they satisfy the Y-system equations

$$Y_s(\xi + \mathrm{i})Y_s(\xi - \mathrm{i}) = [1 + Y_{s+1}(\xi)][1 + Y_{s-1}(\xi)] \qquad s = 1, 2, \ldots. \tag{15}$$

Finally, we introduce the exponential counting function

$$\mathcal{F}(\xi) = \frac{Q(\xi + 2\mathrm{i})}{Q(\xi - 2\mathrm{i})} \frac{T_0(\xi - \mathrm{i})}{T_0(\xi + \mathrm{i})} \tag{16}$$

which can be used to rewrite the Bethe ansatz equations as

$$\mathcal{F}(w_j) = -1 \qquad j = 1, \ldots, m. \tag{17}$$

Combining the above definitions it is easy to see that the Y-functions can simply be expressed in terms of the exponential counting function starting with

$$1 + Y_1(\xi) = [1 + \mathcal{F}(\xi + \mathrm{i})]\left[1 + \frac{1}{\mathcal{F}(\xi - \mathrm{i})}\right] \tag{18}$$

and then using the Y-system equations (15) recursively.

The above algebraic considerations are supplemented by the observation that although the $T_s(\xi)$ as defined by (9) and (10) appear to be trigonometric rational functions, in fact they are trigonometric polynomials. This is a consequence of the Bethe ansatz equations (2). It is also easy to see that

$$T_s(\xi) \sim \mathrm{e}^{\frac{\gamma}{2}N|\xi|} \tag{19}$$

asymptotically for $\xi \to \pm\infty$. The Y-functions are trigonometric rational functions, bounded for large $|\xi|$.

In the rest of this paper we restrict our attention to the following special case of the problem.

- $p_0 = p + 1$ integer ($p \geqslant 2$)
- w_j all real and different
- $\omega = 0$ or $\omega = \frac{\pi}{2}$

We have chosen these restrictions because while they make the discussion technically much easier than for the general case they still contain many physically interesting cases. We will discuss in detail

- case $\mathcal{A}$: $\omega = 0$
- case $\mathcal{B}_1$: $\omega = \frac{\pi}{2}$ $\quad p_0$ odd
- case $\mathcal{B}_2$: $\omega = \frac{\pi}{2}$ $\quad p_0$ even.

For the special case of p_0 integer, the T-functions satisfy

$$T_{p_0}(\xi) - T_{p_0-2}(\xi) = 2(-1)^m \cos(\omega p_0) T_0(\xi + \mathrm{i}p_0) \tag{20}$$

and for cases $\mathcal{A}$ and $\mathcal{B}_2$ this allows us to define

$$K(\xi) = (-1)^m \cos(\omega p_0) \frac{T_{p_0-2}(\xi)}{T_0(\xi + \mathrm{i}p_0)} \tag{21}$$

which, together with $\{Y_s(\xi), s = 1, 2, \ldots, p_0-2\}$, form a finite Y-system of type D_{p_0}, because

$$K(\xi + \mathrm{i})K(\xi - \mathrm{i}) = 1 + Y_{p_0-2}(\xi) \tag{22}$$

$$[1 + K(\xi)]^2 = 1 + Y_{p_0-1}(\xi). \tag{23}$$

For case $\mathcal{B}_1$ (20) implies the reflection symmetry

$$T_{p_0-1-k}(\xi) = T_{p_0-1+k}(\xi) \qquad k = 0, 1, \ldots, p_0 \tag{24}$$

$$Y_{p_0-1-k}(\xi) = Y_{p_0-1+k}(\xi) \qquad k = 0, 1, \ldots, p_0 - 1 \tag{25}$$

and this symmetry allows us to truncate the Y-system after Y_{p_0-1}. We call the corresponding finite Y-system of type $A^s_{2p_0-3}$.

To analyse the Y-system equations further we need some definitions. Let us write

$$N = 2(m + d) \tag{26}$$

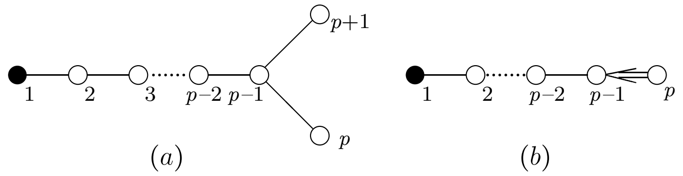

Figure 1. Dynkin diagrams associated with D_{p+1}- and A^s_{2p-1}-type Y-systems.

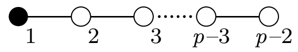

Figure 2. Dynkin diagram associated with A_{p-2}-type Y-systems.

where $0 \leqslant d \leqslant \frac{N}{2}$. A special case of $\mathcal{A}$, for which also $d < p_0$ will play an important role in the following. We call this sub-case $\tilde{\mathcal{A}}$. Similarly we denote by $\tilde{\mathcal{B}}_2$ the sub-case of $\mathcal{B}_2$ with $d < \frac{p_0}{2}$. We next define $\zeta(\xi)$ by

$$\zeta(\xi) = \sum_{j=0}^{p_0-1} q[\xi + \mathrm{i}(2j + 1 - p_0)]. \tag{27}$$

It is easy to see that for cases $\mathcal{A}$ and $\mathcal{B}_2$

$$\zeta(\xi + 2\mathrm{i}) = \zeta(\xi) \tag{28}$$

and

$$T_{p_0-1}(\xi) = \zeta(\xi)Q(\xi + \mathrm{i}p_0)Q(\xi - \mathrm{i}p_0). \tag{29}$$

Further

$$T_{p_0}(\xi) = (-1)^m \kappa T_0(\xi + \mathrm{i}p_0) + \zeta(\xi - \mathrm{i})Q(\xi + \mathrm{i} + \mathrm{i}p_0)Q(\xi - \mathrm{i} - \mathrm{i}p_0) \tag{30}$$

$$T_{p_0-2}(\xi) = (-1)^{m+1} \kappa T_0(\xi + \mathrm{i}p_0) + \zeta(\xi - \mathrm{i})Q(\xi + \mathrm{i} + \mathrm{i}p_0)Q(\xi - \mathrm{i} - \mathrm{i}p_0) \tag{31}$$

where $\kappa = \cos(\omega p_0)$.

Because $\zeta(\xi)$ is periodic with period 2i and is free of singularities in the strip $|\mathrm{Im}\,\xi| \leqslant 1$ as can be seen from (29), it is actually an entire function and thus a Laurent polynomial in the variable $\mathrm{e}^{\pi\xi}$. If $d < p_0$, as is the case for $\tilde{\mathcal{A}}$, then the only such polynomial consistent with the asymptotic equation (19) is $\zeta(\xi) = c_0 = \text{const}$. Similarly for $\tilde{\mathcal{B}}_2$ (19) forces $\zeta(\xi)$ to vanish. In this latter case the D_{p_0}-type Y-system is further reduced since from (29) we have in this case

$$T_{p_0-1}(\xi) = 0 \qquad \text{and} \qquad Y_{p_0-2}(\xi) = 0. \tag{32}$$

The remaining Y-system for $\{Y_s(\xi), s = 1, 2, \ldots, p_0 - 3\}$ is of type A_{p_0-3}.

For later use we note that in case $\tilde{\mathcal{A}}$

$$T_{p_0-1}(\xi) = c_0 Q(\xi + \mathrm{i}p_0)Q(\xi - \mathrm{i}p_0) \tag{33}$$

and this implies that $T_{p_0-1}(\xi)$ has no zeros in the strip $|\mathrm{Im}\,\xi| \leqslant 1$. Similarly in case $\tilde{\mathcal{B}}_2$

$$T_{p_0-2}(\xi) = (-1)^{m+1}(-1)^{\frac{p_0}{2}} T_0(\xi + \mathrm{i}p_0) \tag{34}$$

implies that $T_{p_0-2}(\xi)$ has no zeros in the strip $|\mathrm{Im}\,\xi| \leqslant 1$.

To summarize, for integer $p_0 = p + 1$ we found three different types of finite Y-systems. The Y-functions can be associated with the nodes of the Dynkin diagrams shown in figures 1 and 2. Cases $\mathcal{A}$ and $\mathcal{B}_2$ correspond to the D_{p+1}-type diagram of figure 1(a) and case $\mathcal{B}_1$

corresponds to the A^s_{2p-1}-type diagram of figure 1(*b*). The A_{p-2}-type diagram of figure 2 corresponds to the sub-case $\tilde{\mathcal{B}}_2$. The Y-system equations are of the form

$$W_a(\xi+\mathrm{i})W_a(\xi-\mathrm{i}) = \prod_b [1+W_b(\xi)]^{I_{ab}}. \tag{35}$$

Here $W_a(\xi) = Y_a(\xi)$ for almost all cases with the exception of the last two nodes of the D_{p+1} diagram (figure 1(*a*)). In this case

$$W_{p+1}(\xi) = W_p(\xi) = K(\xi). \tag{36}$$

The matrix elements I_{ab} are defined as follows. I_{ab} is zero if nodes a and b are not connected by links and it is unity if the nodes are connected by a single line. Finally, the oriented double line at the end of the A^s_{2p-1}-type diagram means

$$I_{p-1p} = 1 \qquad I_{pp-1} = 2. \tag{37}$$

Our D_{p+1}- and A_{p-2}-type Y-system equations coincide with the usual Y-system equations [24] associated with the Dynkin diagrams of these simply laced Lie algebras. Our equations corresponding to the diagram of figure 1(*b*) correspond to those solutions of the standard Y-system equations for the Lie algebra A_{2p-1} that are symmetric under reflection of the Dynkin diagram.

4. The counting function

The next step in the Bethe ansatz solution of the model is the definition of the 'counting' function Z_N [16–18], which will play an important role in our considerations. In (16) we have already defined the exponential counting function which can be used to find the positions of Bethe ansatz 'holes', real solutions of (17), different from the Bethe roots:

$$\mathcal{F}(h_\alpha) = -1 \qquad \alpha = 1, \dots, H. \tag{38}$$

To define Z_N properly, we need some more definitions. If the function $f(z)$ is analytic and non-vanishing (ANZ) in some simply connected domain $\mathcal{U}$ then we can define its 'logarithm' $\log f(z)$ as the primitive function of $f'(z)/f(z)$, up to a constant, which can be fixed by giving the value of $\log f(z_0)$ for some $z_0 \in \mathcal{U}$.

Using the above definition we now introduce the functions

$$\phi_r(\xi) = -\mathrm{i}\log\frac{\sinh\frac{\gamma}{2}(\mathrm{i}r-\xi)}{\sinh\frac{\gamma}{2}(\mathrm{i}r+\xi)} \qquad \phi_r(0) = 0 \tag{39}$$

in the strip $|\mathrm{Im}\,\xi| < r$ (for $0 < r < p_0$). For later use we note that

$$\phi_r(\pm\infty) = \pm\pi\frac{p_0 - r}{p_0}. \tag{40}$$

The counting function Z_N is now defined as[1]

$$Z_N\left(\frac{\pi}{2}\xi\right) = \frac{N}{2}\left\{\phi_1\left(\xi - \frac{2\Theta}{\pi}\right) + \phi_1\left(\xi + \frac{2\Theta}{\pi}\right)\right\} - \sum_{j=1}^{m}\phi_2(\xi - w_j). \tag{41}$$

The precise relation between Z_N and the exponential counting function is

$$\mathcal{F}(\xi) = (-1)^\delta\, \mathrm{e}^{\mathrm{i}Z_N(\frac{\pi}{2}\xi)} \tag{42}$$

[1] The $\frac{\pi}{2}$ factor is present in the argument of Z_N because our variable ξ differs by this factor from the usual rapidity variable.

where

$$\delta \equiv m + \frac{2\omega}{\pi} \qquad (\mathrm{mod}\, 2). \tag{43}$$

Using (40) we can calculate

$$Z_N(\pm\infty) = \pm\pi \left(m + 2d - \frac{2d}{p_0} \right). \tag{44}$$

In the following we will assume that the Bethe ansatz solution we are considering is such that the counting function is essentially monotonically increasing. We require that

$$Z_N'\left(\frac{\pi}{2} w_j\right) > 0 \qquad j = 1, \ldots, m \tag{45}$$

$$Z_N'\left(\frac{\pi}{2} h_\alpha\right) > 0 \qquad \alpha = 1, \ldots, H \tag{46}$$

$$Z_N'\left(\frac{\pi}{2} \xi\right) > 0 \qquad \text{for large } |\xi| \tag{47}$$

and call such solutions non-degenerate. We will see that all solutions are non-degenerate for large enough l but we assume that this property of the solutions is also sustained for smaller values of l.

For non-degenerate solutions (44) can be used to obtain a simple formula that gives the number of holes in terms of Bethe ansatz data:

$$\omega = 0: \qquad H = 2d - 2\left[\frac{1}{2} + \frac{d}{p_0}\right] \qquad \delta \equiv m \ (\mathrm{mod}\, 2) \tag{48}$$

$$\omega = \frac{\pi}{2}: \qquad H = 2d - 1 - 2\left[\frac{d}{p_0}\right] \qquad \delta \equiv m + 1 \ (\mathrm{mod}\, 2). \tag{49}$$

(In the above formulae the square brackets stand for integer part.)

Equations (48) and (49) show that the $\omega = 0$ sector originally studied [16, 17] corresponds to states containing an even number of holes, whereas states with an odd number of holes are in the $\omega = \frac{\pi}{2}$ sector. The twist parameter ω has been used in the description of restricted sine–Gordon models [19] but here we see that it can also be used to study the odd particle sector in the original (unrestricted) sine–Gordon and massive Thirring models. The description of this important sector of the theory (containing the physically most interesting one-particle states) has been based on conjectures [18] but as we see here it can be studied on an equal footing with the even sector.

5. Lattice TBA equations

We now translate the Y-system functional relations (35) into TBA integral equations [21, 22]. As explained in the appendix, we need to know the zeros and poles of $W_a(\xi)$ in the strip $|\mathrm{Im}\,\xi| < 1$ together with their asymptotic behaviour. We will call this strip the main strip. From definitions (13) and (21) it is clear that all $W_a(\xi)$ are bounded at infinity and that in the main strip they can have zeros only, there are no poles. It is also clear from these definitions that the zeros of $W_a(\xi)$ are inherited from the neighbouring $T_s(\xi)$ functions.

Let the set of zeros of $T_k(\xi)$ in the main strip be

$$\hat{r}_k = \left\{ \hat{y}_k^{(n)} \right\}_{n=1}^{\hat{R}_k}. \tag{50}$$

The zeros are not necessarily different from each other, for example

$$\hat{r}_0 = \frac{N}{2}\left\{\frac{2\Theta}{\pi}, -\frac{2\Theta}{\pi}\right\} \qquad \hat{R}_0 = N. \tag{51}$$

The zeros of $T_1(\xi)$ are solutions of the equation

$$\mathcal{F}\big(\hat{y}_1^{(n)}\big) = -1 \tag{52}$$

different from the Bethe roots w_j. This set includes the holes defined in (38) and may contain C additional complex 'holes', complex solutions of (52) in the main strip. Thus, $\hat{R}_1 = H + C$.

The set of zeros of $W_a(\xi)$ will be denoted by

$$\hat{q}_a = \big\{\hat{z}_a^{(\alpha)}\big\}_{\alpha=1}^{\hat{Q}_a} \tag{53}$$

and is given by

$$D_{p+1}\colon\ \hat{q}_a = \hat{r}_{a-1} \cup \hat{r}_{a+1} \qquad a = 1, \ldots, p-1 \tag{54}$$

$$\hat{q}_p = \hat{q}_{p+1} = \hat{r}_{p-1} \tag{55}$$

$$A^s_{2p-1}\colon \hat{q}_a = \hat{r}_{a-1} \cup \hat{r}_{a+1} \qquad a = 1, \ldots, p-1 \tag{56}$$

$$\hat{q}_p = \hat{r}_{p-1} \cup \hat{r}_{p+1} = 2\hat{r}_{p-1} \tag{57}$$

$$A_{p-2}\colon\ \hat{q}_a = \hat{r}_{a-1} \cup \hat{r}_{a+1} \qquad a = 1, \ldots, p-3 \tag{58}$$

$$\hat{q}_{p-2} = \hat{r}_{p-3}. \tag{59}$$

We note that in (59) we have used the fact that $\hat{R}_{p-1} = 0$ as established after (34) and that in the special case $\tilde{\mathcal{A}}\ \hat{R}_p = 0$ as follows from (33).

To be able to use the results of the appendix we make the following definitions:

$$l_a(u) = \sum_b I_{ab} \ln[1 + W_b(u)] \tag{60}$$

$$\beta_a(\xi) = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}u \frac{l_a(u)}{\cosh\frac{\pi}{2}(\xi - u)} \tag{61}$$

$$\alpha_a(u) = \frac{1}{4}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}v \frac{l_a(v)}{\sinh\frac{\pi}{2}(v - u)}. \tag{62}$$

(In the last formula $\mathcal{P}$ stands for principal value integration.)

Using the main result (A6) of the appendix we now transform the Y-system equations (35) into TBA integral equations:

$$W_a(\xi) = \hat{\sigma}_a \prod_{\alpha=1}^{\hat{Q}_a} \tau\big(\xi - \hat{z}_a^{(\alpha)}\big) \exp\{\beta_a(\xi)\} \tag{63}$$

where $\hat{\sigma}_a$ is the sign of $W_a(\infty)$ and $\tau(\xi)$ is defined in (A7).

The integral equations (63) are not sufficient to reconstruct the functions $W_a(\xi)$ completely. They have to be supplemented by the 'quantization conditions'

$$1 + W_a\big(\hat{y}_a^{(n)} \pm \mathrm{i}\big) = 0 \qquad n = 1, \ldots, \hat{R}_a \quad a = 1, \ldots, p. \tag{64}$$

($a = 1, \ldots, p-2$ for the A_{p-2} system.)

It is easy to see that (64) follows from the Y-system equations but it is also true that (63) and (64) together are completely equivalent to the Y-system equations (35).

6. Continuum limit

We are interested in the finite volume continuum limit, which is obtained by taking $N \to \infty$ and tuning the inhomogeneity parameter simultaneously according to (8). In the Bethe ansatz language in this limit we have to take $m \to \infty$ while keeping d, H fixed.

We assume that the Y-system functions $W_a(\xi)$ have well-defined continuum limits. The corresponding limit of the set of zeros $\hat{r}_k$ will be denoted by r_k:

$$\hat{r}_k = \left\{\hat{y}_k^{(n)}\right\}_{n=1}^{\hat{R}_k} \quad \longrightarrow \quad r_k = \left\{y_k^{(n)}\right\}_{n=1}^{R_k} \tag{65}$$

where $R_k \leqslant \hat{R}_k$, since some of the zeros can disappear in the process by going to infinity.

Actually, all the zeros of $T_0(\xi)$ go away in the limit and the corresponding factor

$$C_N(\xi) = \prod_{n=1}^{N} \tau\left(\xi - \hat{y}_0^{(n)}\right) = (-1)^{\frac{N}{2}} \left\{ \frac{1 - \frac{l}{2N}\, \mathrm{e}^{\frac{\pi}{2}\xi}}{1 + \frac{l}{2N}\, \mathrm{e}^{\frac{\pi}{2}\xi}} \frac{1 - \frac{l}{2N}\, \mathrm{e}^{-\frac{\pi}{2}\xi}}{1 + \frac{l}{2N}\, \mathrm{e}^{-\frac{\pi}{2}\xi}} \right\}^{\frac{N}{2}} \tag{66}$$

becomes

$$(-1)^{\frac{N}{2}} \exp\left(-l \cosh \frac{\pi}{2}\xi\right) \tag{67}$$

in the continuum limit.

We denote by σ_a the sign of $W_a(\infty)$ in the continuum limit, which can also differ from $\hat{\sigma}_a$.

In the continuum limit

$$\hat{q}_a \quad \longrightarrow \quad q_a = \left\{z_a^{(\alpha)}\right\}_{\alpha=1}^{Q_a} \tag{68}$$

where

$$q_1 = r_2 \qquad q_a = r_{a-1} \cup r_{a+1} \qquad a = 2, \ldots, p-1 \tag{69}$$

($a = 2, \ldots, p-3$ for the A_{p-2} case) and

$$\begin{aligned} q_p &= q_{p+1} = r_{p-1} && (D_{p+1}) \\ q_p &= 2r_{p-1} && \left(A_{2p-1}^s\right) \\ q_{p-2} &= r_{p-3} && (A_{p-2}\, p \geqslant 5). \end{aligned} \tag{70}$$

With these definitions the TBA integral equations in the continuum limit become

$$W_a(\xi) = \sigma_a \exp\left(-\delta_{a1} l \cosh \frac{\pi}{2}\xi\right) \prod_{\alpha=1}^{Q_a} \tau\left(\xi - z_a^{(\alpha)}\right) \exp\{\beta_a(\xi)\} \tag{71}$$

which have to be supplemented by the quantization conditions

$$1 + W_a\left(y_a^{(n)} \pm \mathrm{i}\right) = 0 \qquad n = 1, \ldots, R_a \quad a = 1, \ldots, p. \tag{72}$$

($a = 1, \ldots, p-2$ for the A_{p-2} system.) The exponential factor $\exp\left(-l \cosh \frac{\pi}{2}\xi\right)$ is present in the TBA equation for $a = 1$ only. This is indicated in figures 1 and 2 by colouring the corresponding nodes black.

An important special case is when all zeros are real. In this case the modulus of $W_a\left(y_a^{(n)} \pm \mathrm{i}\right)$ is automatically equal to unity and (72) can be rewritten as

$$(\mathrm{i})^{Q_a} \exp -\mathrm{i} \left\{ \delta_{a1} l \sinh\left(\frac{\pi}{2} y_a^{(n)}\right) - \alpha_a\left(y_a^{(n)}\right) + \sum_{\alpha=1}^{Q_a} \gamma\left(y_a^{(n)} - z_a^{(\alpha)}\right) \right\} = -\sigma_a \tag{73}$$

for $n = 1, \ldots, R_a$ and $a = 1, \ldots, p$ ($a = 1, \ldots, p-2$ for A_{p-2}). In (73) the notation

$$\gamma(u) = 2 \arctan(\tau(u)) \tag{74}$$

is used. Note that $|\gamma(u)| \leqslant \frac{\pi}{2}$ for real u.

7. Energy and momentum

Formulae (7) determining the energy and momentum of the Bethe ansatz state can be rewritten in terms of $T_0(\xi)$ and $T_1(\xi)$ as

$$\mathrm{e}^{-2\mathrm{i}aE} = \frac{T_1\left(\frac{2\Theta}{\pi} - \mathrm{i}\right) T_1\left(\mathrm{i} - \frac{2\Theta}{\pi}\right)}{T_0^2\left(\frac{2\Theta}{\pi} - 2\mathrm{i}\right)} \qquad \mathrm{e}^{-2\mathrm{i}aP} = \mathrm{e}^{-4\mathrm{i}\omega}\frac{T_1\left(\frac{2\Theta}{\pi} - \mathrm{i}\right)}{T_1\left(\mathrm{i} - \frac{2\Theta}{\pi}\right)}. \tag{75}$$

From (14) we see that $T_1(\xi)$ satisfies

$$T_1(\xi + \mathrm{i})T_1(\xi - \mathrm{i}) = [1 + Y_1(\xi)]T_0(\xi + 2\mathrm{i})T_0(\xi - 2\mathrm{i}) \tag{76}$$

which is of the form (A1) and can be solved by the techniques explained in the appendix. For this purpose we first define

$$T_1(\xi) = \hat{\sigma}\tilde{T}_1(\xi)\Phi^{N/2}(\xi) \tag{77}$$

where $\hat{\sigma}$ is a sign, $\tilde{T}_1(\xi)$ has the same zeros as $T_1(\xi)$, has no poles, it is positive for $\xi \to \infty$ and satisfies

$$\tilde{T}_1(\xi + \mathrm{i})\tilde{T}_1(\xi - \mathrm{i}) = 1 + Y_1(\xi). \tag{78}$$

In (77) the factor $\Phi(\xi)^{N/2}$ solves the T_0-dependent part of (76):

$$\Phi(\xi) = \exp\{\Delta(\xi)\} \qquad \Delta(\xi) = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}u \frac{\ln B\left(u, \frac{2\Theta}{\pi}\right)}{\cosh\frac{\pi}{2}(\xi - u)} \tag{79}$$

where

$$B(u, \lambda) = k(u + \lambda + 2\mathrm{i})k(u + \lambda - 2\mathrm{i})k(u - \lambda + 2\mathrm{i})k(u - \lambda - 2\mathrm{i}). \tag{80}$$

Now we take the logarithm of (75) and get

$$E = E_0 + \bar{E} \qquad \bar{E} = S_+ + S_- \tag{81}$$

$$P = P_0 + \bar{P} \qquad \bar{P} = S_+ - S_- \tag{82}$$

where

$$S_+ = \frac{\mathrm{i}}{2a}\ln\tilde{T}_1\left(\frac{2\Theta}{\pi} - \mathrm{i}\right) \qquad S_- = \frac{\mathrm{i}}{2a}\ln\left[(-1)^H\tilde{T}_1\left(\mathrm{i} - \frac{2\Theta}{\pi}\right)\right] \tag{83}$$

and the energy and momentum constants are

$$E_0 = \frac{\pi}{a}N_0 - \frac{\omega}{a} + \frac{\mathrm{i}N}{2a}\left\{\Delta\left(\frac{2\Theta}{\pi} - \mathrm{i}\right) - \frac{\mathrm{i}\pi}{2} + \ln\sin\gamma - \ln\sinh\gamma\left(\frac{2\theta}{\pi} - \mathrm{i}\right)\right\} \tag{84}$$

$$P_0 = \frac{\omega}{a} + \frac{\pi}{a}N_1. \tag{85}$$

Here the choice of the integers N_0 and N_1 (which are in principle arbitrary) is part of the regularization scheme. If we choose $N_0 = N_1 = 0$ we get

$$P_0 = \frac{\omega}{a} \qquad \text{and} \qquad E_0 = -\frac{\omega}{a} + \frac{N}{2a}\chi_0\left(\frac{4\Theta}{\pi}\right). \tag{86}$$

The precise definition of the function $\chi_0(\xi)$ will be given later. What is important here is that E_0 and P_0 are universal in the sense that they are the same for all states in a given sector characterized by $\omega = 0$ or $\omega = \frac{\pi}{2}$. Omitting these unphysical constants leaves us with the physical energy and momentum eigenvalues $\bar{E}$ and $\bar{P}$.

Using the method outlined in the appendix we can solve (78):

$$\tilde{T}_1(\xi) = \prod_{\alpha=1}^{H} \tau(\xi - h_\alpha) \prod_{\beta=1}^{C} \tau(\xi - \Omega_\beta) \exp\left\{\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}u \frac{\ln[1 + Y_1(u)]}{\cosh\frac{\pi}{2}(\xi - u)}\right\}. \quad (87)$$

Here Ω_β are complex solutions of (52). Using definitions (83) finally we get in the continuum limit

$$\bar{E} = \mathcal{M}\left[\sum_{\alpha=1}^{H} \cosh\frac{\pi h_\alpha}{2} + \sum_{\beta=1}^{C} \cosh\frac{\pi\Omega_\beta}{2} - \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}u \cosh\frac{\pi u}{2} \ln[1 + Y_1(u)]\right] \quad (88)$$

$$\bar{P} = \mathcal{M}\left[\sum_{\alpha=1}^{H} \sinh\frac{\pi h_\alpha}{2} + \sum_{\beta=1}^{C} \sinh\frac{\pi\Omega_\beta}{2} - \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}u \sinh\frac{\pi u}{2} \ln[1 + Y_1(u)]\right]. \quad (89)$$

8. The A_1 case

One of the simplest cases is $p = 3$, $d = H = 1$. In this case we can determine the energy and momentum eigenvalues exactly.

In this case $\delta \equiv \frac{N}{2}$ and from (13) it follows that $Y_1(\infty) > 0$. Further $\beta_1(\xi) = 0$ and $\hat{R}_2 = 0$ here and (63) reduces to

$$Y_1(\xi) = W_1(\xi) = C_N(\xi). \quad (90)$$

It is easy to see that there cannot be any complex holes in this case. Thus $\hat{R}_1 = 1$ and we introduce $\hat{y}_1^{(1)} = \hat{h}$ for the position to the single (real) hole. The solution of (64) gives

$$\hat{h} = h + \frac{h_2}{N^2} + \mathcal{O}(N^{-3}) \quad (91)$$

where h is the solution of

$$l \sinh\frac{\pi}{2}h = 2\pi M_1 \qquad M_1 \equiv \frac{1+\delta}{2} (\mathrm{mod}\ 1) \quad (92)$$

and

$$h_2 = \frac{l^2}{6\pi}\frac{\sinh\left(\frac{3}{2}\pi h\right)}{\cosh\left(\frac{1}{2}\pi h\right)}. \quad (93)$$

In the continuum limit we get

$$Y_1(\xi) = (-1)^\delta \exp\left(-l\cosh\frac{\pi}{2}\xi\right) \quad (94)$$

and the energy and momentum eigenvalues are

$$\bar{E} = \mathcal{M}\cosh\frac{\pi h}{2} - \frac{\mathcal{M}}{4}\int_{-\infty}^{\infty} \mathrm{d}u \cosh\frac{\pi u}{2} \ln\left[1 + (-1)^\delta \exp\left(-l\cosh\frac{\pi}{2}u\right)\right] \quad (95)$$

$$\bar{P} = \mathcal{M}\sinh\frac{\pi h}{2} = \frac{2\pi M_1}{L}. \quad (96)$$

It is interesting to note that the momentum eigenvalues (96) are exactly the same as they would be in a free theory. The $p = 3$ model is not free, however, and the energy eigenvalues differ from those of the free theory by the second term in (95). On the other hand, this shift is independent of the momentum quantum number M_1.

9. Destri–deVega equation

In the sine–Gordon (MT) model a non-linear integral equation (Destri–deVega equation) is used to describe the Bethe ansatz states and calculate their energy and momentum [16–18]. This is formulated in terms of the counting function Z_N and its exponential, $\mathcal{F}(\xi)$.

Let us choose the parameter B so that there are no complex holes in the strip $0 < \mathrm{Im}\,\xi < B$. Then we can define

$$\mathcal{L}_+(\xi) = \log[1 + \mathcal{F}(\xi)] \qquad 0 < \mathrm{Im}\,\xi < B. \tag{97}$$

We define further the function $\chi(\xi)$ by

$$\chi'(\xi) = G(\xi) \qquad \chi(0) = 0 \tag{98}$$

where

$$G(\xi) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k\xi} \frac{\sinh(p-1)k}{2\cosh k \sinh pk}. \tag{99}$$

The function $\chi_0(\xi)$ occurring in (86) is defined analogously with p replaced by $p_0 = p + 1$.

For non-degenerate states

$$\mathcal{L}_+(\xi) = \hat{L}_+(\xi) = \ln\left[1 + (-1)^\delta \exp\left(\mathrm{i}Z_N\left(\frac{\pi}{2}\xi\right)\right)\right] \tag{100}$$

and the DdV equation can be written as

$$\begin{aligned} Z_N\left(\frac{\pi}{2}\xi\right) = {} & \frac{N}{2}\left\{\arctan\left[\sinh\left(\frac{\pi}{2}\xi - \Theta\right)\right] + \arctan\left[\sinh\left(\frac{\pi}{2}\xi + \Theta\right)\right]\right\} + \sum_{\alpha=1}^{H} \chi(\xi - h_\alpha) \\ & + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}u [G(\xi - u - \mathrm{i}\eta)\hat{L}_+(u + \mathrm{i}\eta) - G(\xi - u + \mathrm{i}\eta)\hat{L}_+^*(u + \mathrm{i}\eta)] \end{aligned} \tag{101}$$

where $0 < \eta < B$. Analogously to the quantization conditions (64) supplementing the TBA integral equations, the DdV equations are supplemented by the quantization conditions

$$Z_N\left(\frac{\pi}{2}h_\alpha\right) = 2\pi M_\alpha \qquad M_\alpha \equiv \frac{1+\delta}{2} (\mathrm{mod}\, 1) \qquad \alpha = 1, \ldots, H. \tag{102}$$

The counting function Z_N has the continuum limit Z in terms of which the continuum DdV equation reads

$$\begin{aligned} Z\left(\frac{\pi}{2}\xi\right) = {} & l \sinh \frac{\pi}{2}\xi + \sum_{\alpha=1}^{H} \chi(\xi - h_\alpha) \\ & + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}u [G(\xi - u - \mathrm{i}\eta)L_+(u + \mathrm{i}\eta) - G(\xi - u + \mathrm{i}\eta)L_+^*(u + \mathrm{i}\eta)] \end{aligned} \tag{103}$$

where

$$L_+(\xi) = \ln\left[1 + (-1)^\delta \exp\left(\mathrm{i}Z\left(\frac{\pi}{2}\xi\right)\right)\right]. \tag{104}$$

The energy and momentum in the continuum can be expressed in terms of the positions of the holes h_α and L_+:

$$\bar{E} = \mathcal{M} \sum_{\alpha=1}^{H} \cosh \frac{\pi h_\alpha}{2} - \frac{\mathcal{M}}{2} \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}u \sinh \frac{\pi}{2}(u + \mathrm{i}\eta) L_+(u + \mathrm{i}\eta) \tag{105}$$

$$\bar{P} = \mathcal{M} \sum_{\alpha=1}^{H} \sinh \frac{\pi h_\alpha}{2} - \frac{\mathcal{M}}{2} \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}u \cosh \frac{\pi}{2}(u + \mathrm{i}\eta) L_+(u + \mathrm{i}\eta). \tag{106}$$

For large physical size l the function L_+ is exponentially small and the integrals containing it in the DdV equation (103) and also in the energy and momentum expressions can be neglected. In this approximation, the quantization conditions (102) are nothing but the (dressed) Bethe ansatz equations for solitons/fermions in the SG/MT model. This leads to the identification

$$\text{SG: (soliton–soliton)} \qquad S(\theta) = -\exp\left\{ \mathrm{i}\chi\left(\frac{2\theta}{\pi}\right)\right\} \qquad \delta \equiv H \pmod 2 \tag{107}$$

and

$$\text{MT: (fermion–fermion)} \qquad S(\theta) = \exp\left\{ \mathrm{i}\chi\left(\frac{2\theta}{\pi}\right)\right\} \qquad \delta = 0 \tag{108}$$

where $S(\theta)$ is the scattering phase for soliton/fermion scattering.

The parameter p is related to the usual SG coupling β by

$$\beta^2 = \frac{8\pi p}{p+1}. \tag{109}$$

$p = 1$ corresponds to the free fermion point and $p = \infty$ to the XY model (asymptotically free point).

For large l the first term on the right-hand side of (103) dominates and this shows that $Z\left(\frac{\pi}{2}\xi\right)$ is monotonically increasing for real ξ. We already used the assumption that this property remains valid also for smaller l.

The DdV equation (103) is suitable for numerical calculations. To study the analytic properties of the solution it is useful to rewrite it in the form

$$Z\left(\frac{\pi}{2}\xi\right) = l\sinh\frac{\pi}{2}\xi + \sum_{\alpha=1}^{H}\chi(\xi - h_\alpha) + \int_{-\infty}^{\infty}\mathrm{d}u\, G(\xi - u)Q(u) \tag{110}$$

where

$$Q(u) = \frac{1}{\pi}\lim_{\eta\to 0}\operatorname{Im} L_+(u + \mathrm{i}\eta). \tag{111}$$

The representation (110) is valid in the strip $|\operatorname{Im}\xi| < 2$. Note that we are considering only the repulsive case $p > 1$ in this paper. In the attractive regime $p < 1$ some of our expressions have to be modified.

We now analytically continue the counting function to the whole complex ξ plane. For this purpose we define

$$\gamma_1(\xi) = -\mathrm{i}\log w(\xi) \qquad \gamma_1(\mathrm{i}p) = \pi \qquad 1 < \operatorname{Im}\xi < 2p - 1 \tag{112}$$

$$\gamma_2(\xi) = -\mathrm{i}\log w(\xi) \qquad \gamma_2(-\mathrm{i}p) = \pi \qquad -2p + 1 < \operatorname{Im}\xi < -1 \tag{113}$$

where

$$w(\xi) = \frac{\sinh\frac{\pi}{2p}(\mathrm{i} - \xi)}{\sinh\frac{\pi}{2p}(\mathrm{i} + \xi)}. \tag{114}$$

The analytical continuation of the counting function in the strip $2 < \operatorname{Im}\xi < 2p$ is

$$Z_1\left(\frac{\pi}{2}\xi\right) = \delta\pi + \sum_{\alpha=1}^{H}\gamma_1(\xi - \mathrm{i} - h_\alpha) + \int_{-\infty}^{\infty}\mathrm{d}u\, K_1(\xi - u)Q(u) \tag{115}$$

where

$$K_1(\xi) = G(\xi) + G(\xi - 2\mathrm{i}) \tag{116}$$

which is analytic in this strip.

Similarly in the strip $-2p < \mathrm{Im}\,\xi < -2$ the analytical continuation is

$$Z_2\left(\frac{\pi}{2}\xi\right) = \delta\pi + \sum_{\alpha=1}^{H} \gamma_2(\xi + \mathrm{i} - h_\alpha) + \int_{-\infty}^{\infty} \mathrm{d}u\, K_2(\xi - u)Q(u) \tag{117}$$

where

$$K_2(\xi) = G(\xi) + G(\xi + 2\mathrm{i}) \tag{118}$$

which is analytic in this strip.

The exponential counting function is thus defined as

$$\mathcal{F}(\xi) = \begin{cases} (-1)^\delta \exp\left(\mathrm{i}Z_1\left(\frac{\pi}{2}\xi\right)\right) & 2 < \mathrm{Im}\,\xi < 2p \\ (-1)^\delta \exp\left(\mathrm{i}Z\left(\frac{\pi}{2}\xi\right)\right) & |\mathrm{Im}\,\xi| < 2 \\ (-1)^\delta \exp\left(\mathrm{i}Z_2\left(\frac{\pi}{2}\xi\right)\right) & -2p < \mathrm{Im}\,\xi < -2 \end{cases} \tag{119}$$

and it is easy to see that this defines a meromorphic function with properties

$$\mathcal{F}(\xi + 2\mathrm{i}p_0) = \mathcal{F}(\xi) \qquad \mathcal{F}^*(\xi) = \frac{1}{\mathcal{F}(\xi^*)}. \tag{120}$$

It has poles (∞ many) at $\xi = w_j + 2\mathrm{i}$ and zeros (∞ many) at $\xi = w_j - 2\mathrm{i}$.

10. Infinite volume solution

We have seen that for large l the integrals give exponentially small contribution to the counting function Z. This is also true for Z_1 and Z_2 although this cannot be seen immediately from (115) and (117). However, it is possible to write Z_1 and Z_2 in a form analogous to (103) and show that the corresponding integrals are indeed exponentially small. Neglecting the integrals we have

$$\mathcal{F}(\xi) = [1 + \mathcal{O}(\mathrm{e}^{-\mathcal{K}l})] \cdot \begin{cases} u(\xi)u(\xi - 2\mathrm{i}) & 2 < \mathrm{Im}\,\xi < 2p \\ \exp\left(\mathrm{i}l \sinh \frac{\pi}{2}\xi\right) u(\xi) & |\mathrm{Im}\,\xi| < 2 \\ u(\xi)u(\xi + 2\mathrm{i}) & -2p < \mathrm{Im}\,\xi < -2 \end{cases} \tag{121}$$

where $\mathcal{K}$ is $\mathrm{Im}\,\xi$-dependent and positive. The function $u(\xi)$ is defined by

$$u(\xi) = (-1)^\delta \prod_{\alpha=1}^{H} \sigma(\xi - h_\alpha) \tag{122}$$

where

$$\sigma(\xi) = \mathrm{e}^{\mathrm{i}\chi(\xi)} \tag{123}$$

is meromorphic with poles and zeros on the imaginary axis and satisfies

$$\sigma(\xi + \mathrm{i})\sigma(\xi - \mathrm{i}) = w(\xi). \tag{124}$$

Since we can express the Y-system functions in terms of the counting function starting with (18) we are able to also calculate the Y-functions with exponential precision for large l. For $Y_1(\xi)$ we find

$$Y_1(\xi) = [1 + \mathcal{O}(\mathrm{e}^{-\mathcal{K}l})] \cdot \begin{cases} \eta_2(\xi - \mathrm{i}) & 3 < \mathrm{Im}\,\xi < 2p - 1 \\ \lambda(\xi) \exp\left(-l \cosh \frac{\pi}{2}\xi\right) & |\mathrm{Im}\,\xi| < 3 \\ \eta_2(\xi + \mathrm{i}) & -2p + 1 < \mathrm{Im}\,\xi < -3 \end{cases} \tag{125}$$

and using the Y-system equations (15) recursively we get for $k = 2, \ldots, p$

$$Y_k(\xi) = [1 + \mathcal{O}(\mathrm{e}^{-\mathcal{K}l})]\eta_k(\xi) \qquad |\mathrm{Im}\,\xi| < k. \tag{126}$$

In (125)

$$\lambda(\xi) = u(\xi + \mathrm{i}) + \frac{1}{u(\xi - \mathrm{i})} \tag{127}$$

and

$$\eta_2(\xi) = \lambda(\xi + \mathrm{i})\lambda(\xi - \mathrm{i}) - 1. \tag{128}$$

The functions $\eta_k(\xi)$ satisfy the Y-system equations

$$\eta_k(\xi + \mathrm{i})\eta_k(\xi - \mathrm{i}) = [1 + \eta_{k+1}(\xi)][1 + \eta_{k-1}(\xi)] \tag{129}$$

for $k = 2, 3, \ldots$ and this determines[2] $\eta_k(\xi)$ for $k > 2$.

It is possible to find the solution of (129) explicitly. We note first that there is a class of solutions depending on a parameter q and a function $B(\xi)$. Using these input data we first define

$$t_0(\xi) = 0 \qquad t_k(\xi) = \sum_{j=0}^{k-1} q^j B[\xi + \mathrm{i}(k - 1 - 2j)] \qquad k = 1, 2 \ldots \tag{130}$$

and then it is easy to show that

$$\eta_k(\xi) = q^{1-k} \frac{t_{k+1}(\xi)t_{k-1}(\xi)}{B(\xi + \mathrm{i}k)B(\xi - \mathrm{i}k)} \qquad k = 1, 2, \ldots \tag{131}$$

solves (129). Equation (131) is quite analogous to (13) and it is also true that

$$1 + \eta_k(\xi) = q^{1-k} \frac{t_k(\xi + \mathrm{i})t_k(\xi - \mathrm{i})}{B(\xi + \mathrm{i}k)B(\xi - \mathrm{i}k)} \qquad k = 1, 2, \ldots \tag{132}$$

The actual solution entering (125) and (126) corresponds to the choice

$$q = (-1)^H \qquad B(\xi) = \prod_{\alpha=1}^{H} \sinh \frac{\pi}{2p}(\xi - h_\alpha). \tag{133}$$

This can be verified by computing $\eta_2(\xi)$ from (128) and from (131) with the above choice and showing that they coincide.

Similarly to the general Y-system equations, the set of η-functions can also be truncated. In cases $\mathcal{A}$ and $\mathcal{B}_2$ we can define

$$\kappa(\xi) = \frac{t_{p-1}(\xi)}{B(\xi - \mathrm{i}p)} \tag{134}$$

and then the relations

$$\kappa(\xi + \mathrm{i})\kappa(\xi - \mathrm{i}) = 1 + \eta_{p-1}(\xi) \qquad 1 + \eta_p(\xi) = [1 + \kappa(\xi)]^2 \tag{135}$$

can be used to show that the truncated system is of type D_{p+1}. If $H < p$ is also satisfied ($\tilde{\mathcal{B}}_2$ case) then $t_p(\xi) = 0$ implies $\eta_{p-1}(\xi) = 0$ and this corresponds to the further truncation to A_{p-2}. Finally, in the $\mathcal{B}_1$ case reflection relations analogous to (24) and (25) are satisfied by $t_k(\xi)$ and $\eta_k(\xi)$ and the truncation is of type A^s_{2p-1} accordingly.

Similarly to the full Y-system where the zeros are determined by those of the T-system, the zeros of $t_k(\xi)$ determine the zeros of $\eta_k(\xi)$. More concretely,

$$\begin{array}{lll} \lambda(\xi)\colon \mathrm{ANZ} & |\mathrm{Im}\,\xi| < 3 & \text{except for zeros of } t_2(\xi) \\ \eta_k(\xi)\colon \mathrm{ANZ} & |\mathrm{Im}\,\xi| < k & \text{except for zeros of } t_{k+1}(\xi)t_{k-1}(\xi) \\ \kappa(\xi)\colon \mathrm{ANZ} & |\mathrm{Im}\,\xi| < p & \text{except for zeros of } t_{p-1}(\xi). \end{array} \tag{136}$$

[2] $\eta_1(\xi) = 0$ by definition.

For later use we note that

$$\lambda(\pm\infty) = 2(-1)^\delta \cos\left[\frac{\pi H}{2}\left(1 - \frac{1}{p}\right)\right]. \tag{137}$$

We now consider some simple examples.

$\boxed{H = 0}$

In this case we have a D_{p+1} system for $p \geqslant 2$. From (122) and (127) we get

$$\lambda(\xi) = 2(-1)^\delta \tag{138}$$

where, according to (107) and (108), only the choice $\delta = 0$ is physical. Further

$$B(\xi) = 1 \qquad t_k(\xi) = k \qquad \eta_k(\xi) = k^2 - 1 \qquad \kappa(\xi) = p - 1. \tag{139}$$

$\boxed{H = 1}$

Here, according to (107) and (108), we have two choices:

$$\delta = 1 \qquad \text{for} \quad \text{SG} \qquad \delta = 0 \qquad \text{for} \quad \text{MT} \tag{140}$$

and we have an A^s_{2p-1} system for $p \geqslant 2$ even and an A_{p-2} system for $p \geqslant 3$ odd. The position of the hole is denoted by $h_1 = h$. From (122) and (127) we find in this case

$$\lambda(\xi) = (-1)^\delta\{\exp(\mathrm{i}\chi(\xi - h + \mathrm{i})) + \exp(-\mathrm{i}\chi(\xi - h - \mathrm{i}))\} \tag{141}$$

and from

$$B(\xi) = \sinh\frac{\pi}{2p}(\xi - h) \tag{142}$$

we have

$$t_k(\xi) = \begin{cases} \dfrac{\cos\left(\frac{k\pi}{2p}\right)}{\cos\left(\frac{\pi}{2p}\right)} \sinh\dfrac{\pi(\xi - h)}{2p} & k \text{ odd} \\ \mathrm{i}\dfrac{\sin\left(\frac{k\pi}{2p}\right)}{\cos\left(\frac{\pi}{2p}\right)} \cosh\dfrac{\pi(\xi - h)}{2p} & k \text{ even} \end{cases} \tag{143}$$

$\boxed{H = 2}$

Here we have a D_{p+1} system for $p \geqslant 2$ again and only the choice $\delta = 0$ is physical. For simplicity we restrict our attention to the symmetric case

$$h_1 = h \qquad h_2 = -h \tag{144}$$

which corresponds to

$$B(\xi) = \frac{1}{2}\cosh\frac{\pi\xi}{p} - \frac{1}{2}\cosh\frac{\pi h}{p}. \tag{145}$$

Now we find

$$\lambda(\xi) = \{\exp[\mathrm{i}\chi(\xi - h + \mathrm{i}) + \mathrm{i}\chi(\xi + h + \mathrm{i})] + \exp[-\mathrm{i}\chi(\xi - h - \mathrm{i}) - \mathrm{i}\chi(\xi + h - \mathrm{i})]\} \tag{146}$$

and

$$t_k(\xi) = \frac{1}{2}\frac{\sin\left(\frac{k\pi}{p}\right)}{\sin\left(\frac{\pi}{p}\right)}\cosh\frac{\pi\xi}{p} - \frac{k}{2}\cosh\frac{\pi h}{p}. \tag{147}$$

11. Ground state

The ground state of the system ($H = 0$) belongs to the $\tilde{A}$ case and corresponds to a D_{p+1}-type Y-system for $p \geqslant 2$. The only physical choice is $\delta = 0$. From (139) we see that for $l \to \infty$ there are no zeros and all $W_a(\xi)$ functions are positive. Our assumption is that these qualitative properties of the solution remain valid also for finite l values. If this is true then (71) becomes

$$W_a(\xi) = \exp\left(-\delta_{a1} l \cosh \frac{\pi}{2}\xi\right) \exp\{\beta_a(\xi)\} \qquad a = 1, \ldots, p+1 \tag{148}$$

and there are no quantization conditions in this case. By taking the logarithm on both sides of (148) we reproduce the customary TBA system describing the ground state of the SG/MT model. Noting that both $W_a(\xi)$ and $\beta_a(\xi)$ are now even functions we find from (88) and (89)

$$P^{(0)} = 0 \qquad E^{(0)} = -\frac{\mathcal{M}}{4}\int_{-\infty}^{\infty} \mathrm{d}u \cosh\frac{\pi u}{2} \ln[1 + W_1(u)]. \tag{149}$$

12. One-particle states

For one-particle states both $\delta = 1$ (SG model) and $\delta = 0$ (MT model) are possible. Moreover, depending on the parity of p, we have an A^s_{2p-1} Y-system ($p \geqslant 2$ even) or A_{p-2} Y-system ($p \geqslant 5$ odd[3]). Here we only consider the SG model with $h_1 = 0$ in detail, because in this case the quantization conditions are satisfied automatically, which simplifies the problem enormously.

Our main assumption will be again that the qualitative features of the $l \to \infty$ solution are also valid for finite l. From (141) and (143) we see that

- $W_a(\xi)$: no zeros for a odd
  double zero at $\xi = 0$ for a even
- all $W_a(\xi)$ are negative, except $W_p(\xi)$ in the A^s_{2p-1} case

and this leads to the TBA equations

$$W_a(\xi) = (-1)^{1+\delta_{pa}} \exp\left(-\delta_{a1} l \cosh\frac{\pi}{2}\xi\right) [\tau(\xi)]^{1+(-1)^a} \exp\{\beta_a(\xi)\} \qquad a = 1, \ldots, p. \tag{150}$$

($a = 1, \ldots, p-2$ for A_{p-2}.) $W_a(\xi)$ and $\beta_a(\xi)$ are even functions and $\alpha_a(\xi)$ are odd. It follows that the quantization conditions (73) are automatically satisfied.

Finally from (88) and (89) we get

$$P^{(1)} = 0 \qquad E^{(1)} = \mathcal{M} - \frac{\mathcal{M}}{4}\int_{-\infty}^{\infty} \mathrm{d}u \cosh\frac{\pi u}{2} \ln[1 + W_1(u)]. \tag{151}$$

13. Two-particle states

The two-particle states are the only ones among our examples where the quantization conditions (73) give non-trivial constraints. The $H = 2$ states belong to the $\tilde{A}$ case and correspond to a D_{p+1}-type Y-system ($p \geqslant 2$) and $\delta = 0$.

Our conjecture is again based on the large l solution. For simplicity, we consider only the following symmetric distribution of zeros:

$$r_k = \{H_k, -H_k\} \qquad k = 1, \ldots, p-1 \tag{152}$$

[3] We have already discussed the A_1 case.

where by convention $H_k > 0$, the positions of the holes are $h_1 = H_1$ and $h_2 = -H_1$ and the signs at infinity are

$$\sigma_1 = -1 \qquad \sigma_a = 1 \quad a = 2, \ldots, p-2 \qquad \sigma_{p-1} = \sigma_p = \sigma_{p+1} = -1. \tag{153}$$

It is important to note that (153) and all subsequent formulae are valid for $p \geqslant 3$ only. The important special case $p = 2$ will be discussed at the end of this section separately.

The TBA integral equations are

$$W_1(\xi) = -\exp\left(-l\cosh\frac{\pi}{2}\xi\right)\tau(\xi - H_2)\tau(\xi + H_2)\exp\{\beta_1(\xi)\} \tag{154}$$

$$W_a(\xi) = \tau(\xi - H_{a-1})\tau(\xi + H_{a-1})\tau(\xi - H_{a+1})\tau(\xi + H_{a+1})\exp\{\beta_a(\xi)\} \tag{155}$$

$$W_{p-1}(\xi) = -\tau(\xi - H_{p-2})\tau(\xi + H_{p-2})\exp\{\beta_{p-1}(\xi)\} \tag{156}$$

$$W_p(\xi) = W_{p+1}(\xi) = -\tau(\xi - H_{p-1})\tau(\xi + H_{p-1})\exp\{\beta_p(\xi)\}. \tag{157}$$

In (155) $a = 2, \ldots, p-2$ and because of the symmetric distribution of zeros the functions $W_a(\xi)$ and $\beta_a(\xi)$ are even and $\alpha_a(\xi)$ are odd. For the same reason it is sufficient to require that (73) are satisfied for the positive zeros:

$$l\sinh\frac{\pi}{2}H_1 + \gamma(H_1 - H_2) + \gamma(H_1 + H_2) - \alpha_1(H_1) = 2\pi M_1 \tag{158}$$

$$\gamma(H_a - H_{a-1}) + \gamma(H_a + H_{a-1}) + \gamma(H_a - H_{a+1}) + \gamma(H_a + H_{a+1}) - \alpha_a(H_a) = 2\pi M_a \tag{159}$$

$$\gamma(H_{p-1} - H_{p-2}) + \gamma(H_{p-1} + H_{p-2}) - \alpha_{p-1}(H_{p-1}) = 2\pi M_{p-1}. \tag{160}$$

In (159) $a = 2, \ldots, p-2$ and all the quantum numbers M_a are half-integers. The value of M_1 is part of the input data, whereas all others (M_a for $a = 2, \ldots, p-1$) are determined from consistency. Numerically we found in all cases we considered that $M_a = 1/2$ for $a = 2, \ldots, p-1$. Finally, the two-particle momentum and energy are

$$P^{(2)} = 0 \qquad E^{(2)} = 2\mathcal{M}\cosh\frac{\pi}{2}H_1 - \frac{\mathcal{M}}{4}\int_{-\infty}^{\infty}\mathrm{d}u\cosh\frac{\pi u}{2}\ln[1 + W_1(u)]. \tag{161}$$

We end this section by discussion of the special case $p = 2$ corresponding to the supersymmetric point of the SG model. Here the TBA equations are

$$W_1(\xi) = \exp\left(-l\cosh\frac{\pi}{2}\xi\right)\exp\{\beta_1(\xi)\} \tag{162}$$

$$W_2(\xi) = W_3(\xi) = -\tau(\xi - H_1)\tau(\xi + H_1)\exp\{\beta_2(\xi)\} \tag{163}$$

and there is just one quantization condition:

$$l\sinh\frac{\pi}{2}H_1 - \alpha_1(H_1) = 2\pi M_1. \tag{164}$$

14. Numerical iteration of the TBA equations

We have calculated the energy and momentum eigenvalues of the ground state and also of one- and two-particle states from the TBA equations numerically and compared the results with those obtained using the DdV equations. We have found agreement to many digits in all cases we considered and concluded that our TBA equations are completely equivalent to the DdV equations.

Our numerical checks covered a rather wide range of the physical size l (from $\mathcal{O}(1)$ down to $l \sim 10^{-2}$) and we considered a number of different choices for the parameter p, including the limit $p = \infty$. The agreement between TBA and DdV is especially spectacular for the A_1 case where we have exact results on the TBA side.

The infinite volume solution we found in section 10, which we have used to help postulating qualitative properties of the Y-system elements, also serves as the starting point of the numerical iteration procedure. We found that the convergence of the numerical iteration is rather fast and already the starting point is a surprisingly good approximation not only for large volumes but also for smaller l values. In the $H = 2$ case this is true in particular for the positions of zeros, which change very little during the iteration.

15. Summary

In this paper we proposed TBA systems describing multi-soliton states of the sine–Gordon (massive Thirring) model. We derived the TBA equations starting from the Bethe ansatz solution of the light-cone lattice formulation of the model, which allows a uniform treatment of states with even and odd soliton numbers. The derivation also provides a simple link between the counting function of the Destri–deVega equation and the Y-system component functions of the TBA approach.

After considering the continuum limit of the lattice model, the only input we need is the distribution of zeros of the TBA Y-functions (and their sign at infinity). This information can be read from the infinite volume solution of the problem, which we found explicitly. Our main assumption in this paper is that what we find for the infinite volume solution remains qualitatively valid for all physical volumes. We verified this assumption numerically by comparing the TBA results with those obtained by the alternative Destri–deVega non-linear integral equation.

The simple pattern of the excited state TBA systems found here gives us some hope that it will be possible to find TBA equations for excited states also in other models, even if no DdV type alternative is available there.

Acknowledgment

This investigation was supported in part by the Hungarian National Science Fund OTKA (under T030099, T034299 and T043159).

Appendix

In this appendix we recall the solution [21, 22] of the basic functional equation

$$f(\xi + \mathrm{i}) f(\xi - \mathrm{i}) = B(\xi). \tag{A1}$$

Here $f(\xi)$ is assumed to be meromorphic in the strip $|\mathrm{Im}\,\xi| < 1 + \epsilon$, satisfying the reality condition

$$f^*(\xi) = f(\xi^*). \tag{A2}$$

Let $\{r_\alpha\}_{\alpha=1}^{R}$ be the set of zeros of $f(\xi)$ with $|\mathrm{Im}\,r_\alpha| < 1$ (not necessarily different from each other) and similarly $\{p_\beta\}_{\beta=1}^{P}$ the set of (not necessarily different) poles with $|\mathrm{Im}\,p_\beta| < 1$. Moreover, for large (real, positive) ξ

$$f(\xi) \sim \hat{\sigma} f_0 \xi^{c_1} \exp(c_0 \xi) \exp\left(-\lambda \cosh \frac{\pi}{2}\xi\right) \tag{A3}$$

asymptotically. Here $\hat{\sigma} = \pm 1$ is a sign, $f_0 > 0$, c_0, c_1 and λ are real.

The function $B(\xi)$ is real analytic in the strip $|\mathrm{Im}\,\xi| < \epsilon$ and non-negative for real ξ. Its asymptotics is given by

$$B(\xi) \sim B_0 \xi^{b_1} \, \mathrm{e}^{b_0 \xi} \tag{A4}$$

where $B_0 > 0$ and b_0, b_1 are real. Comparing (A3) with (A4) gives

$$f_0 = \sqrt{B_0} \qquad c_1 = \frac{b_1}{2} \qquad c_0 = \frac{b_0}{2}. \tag{A5}$$

The key observation that allows the transformation of the Y-system functional relations, which are of the form (A1), to TBA integral equations is the fact that the solution of (A1), satisfying all the above requirements, is uniquely given by

$$f(\xi) = \hat{\sigma} \exp\left(-\lambda \cosh \frac{\pi}{2}\xi\right) \frac{\prod_{\alpha=1}^{R} \tau(\xi - r_\alpha)}{\prod_{\beta=1}^{P} \tau(\xi - p_\beta)} \exp\{\beta(\xi)\} \tag{A6}$$

where

$$\tau(\xi) = \tanh\left(\frac{\pi}{4}\xi\right) \tag{A7}$$

and

$$\beta(\xi) = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}u \frac{\ln B(u)}{\cosh \frac{\pi}{2}(\xi - u)}. \tag{A8}$$

$\beta(\xi)$ is clearly analytic in the strip $|\mathrm{Im}\,\xi| < 1$ and for real u (when $B(u) > 0$)

$$\beta(u \pm \mathrm{i}) = \frac{1}{2}\ln B(u) \pm \frac{\mathrm{i}}{4}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}v \frac{\ln B(v)}{\sinh \frac{\pi}{2}(v - u)}. \tag{A9}$$

Here $\mathcal{P}$ indicates principal value integration.

References

[1] Lüscher M 1986 *Commun. Math. Phys.* **104** 177
Lüscher M 1986 *Commun. Math. Phys.* **105** 153
[2] Klassen T R and Melzer E 1991 *Nucl. Phys.* B **362** 329
[3] Lüscher M, Weisz P and Wolff U 1991 *Nucl. Phys.* B **359** 221
[4] Yang C N and Yang C P 1969 *J. Math. Phys.* **10** 1115
[5] Takahashi M and Suzuki M 1972 *Prog. Theor. Phys.* **48** 2187
[6] Zamolodchikov Al B 1990 *Nucl. Phys.* B **342** 695
[7] Klassen T R and Melzer E 1990 *Nucl. Phys.* B **338** 485
Klassen T R and Melzer E 1991 *Nucl. Phys.* B **350** 635
Zamolodchikov Al B 1991 *Nucl. Phys.* B **358** 497
Zamolodchikov Al B 1991 *Nucl. Phys.* B **366** 122
Fateev V A and Zamolodchikov Al B 1991 *Phys. Lett.* B **271** 91
Ravanini F, Tateo R and Valleriani A 1993 *Int. J. Mod. Phys.* A **8** 1707
Tateo R 1995 *Phys. Lett.* B **355** 157
Fendley P 1999 *Phys. Rev. Lett.* **83** 4468
[8] Klümper A and Pearce P A 1991 *J. Stat. Phys.* **64** 13
Klümper A and Pearce P A 1992 *Physica* A **183** 304
Klümper A, Batchelor M T and Pearce P A 1991 *J. Phys. A: Math. Gen.* **24** 3111
[9] Pearce P A, Chim L and Ahn C 2001 *Nucl. Phys.* B **601** 539
Pearce P A, Chim L and Ahn C 2003 *Nucl. Phys.* B **660** 579
[10] Fendley P 1992 *Nucl. Phys.* B **374** 667
[11] Martins M J 1991 *Phys. Rev. Lett.* **67** 419
[12] Dorey P and Tateo R 1996 *Nucl. Phys.* B **482** 639
Dorey P and Tateo R 1998 *Nucl. Phys.* B **515** 575
[13] Fendley P 1998 *Adv. Theor. Math. Phys.* **1** 210

[14] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1996 *Commun. Math. Phys.* **177** 381
Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 *Commun. Math. Phys.* **190** 247
Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1999 *Commun. Math. Phys.* **200** 297
Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 *Nucl. Phys.* B **489** 487
[15] Ellem R M and Bazhanov V V 2002 *Nucl. Phys.* B **647** 404
[16] Destri C and de Vega H J 1992 *Phys. Rev. Lett.* **69** 2313
Destri C and de Vega H J 1995 *Nucl. Phys.* B **438** 413
[17] Destri C and de Vega H J 1997 *Nucl. Phys.* B **504** 621
Feverati G, Ravanini F and Takács G 1998 *Phys. Lett.* B **430** 264
Feverati G, Ravanini F and Takács G 1999 *Nucl. Phys.* B **540** 543
[18] Feverati G, Ravanini F and Takács G 1998 *Phys. Lett.* B **444** 442
[19] Fioravanti D, Mariottini A, Quattrini E and Ravanini F 1997 *Phys. Lett.* B **390** 243
Feverati G, Ravanini F and Takács G 2000 *Nucl. Phys.* B **570** 615
[20] Zinn-Justin P 1998 *J. Phys. A: Math. Gen.* **31** 6747
[21] Jüttner G, Klümper A and Suzuki J 1998 *Nucl. Phys.* B **512** 581
[22] Kuniba A, Sakai K and Suzuki J 1998 *Nucl. Phys.* B **525** 597
[23] Destri C and de Vega H J 1987 *Nucl. Phys.* B **290** 363
Destri C and de Vega H J 1989 *J. Phys. A: Math. Gen.* **22** 1329
[24] Zamolodchikov Al B 1991 *Phys. Lett.* B **253** 391
Quattrini E, Ravanini F and Tateo R 1995 *Quantum Field Theory and String Theory NATO ASI Series B: Physics* vol 328 p 273